Signals and Systems

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For Her

Contents

Chapter 1

Introduction to Signals and Systems

1.1 Signals

A *signal* is a time-varying physical phenomenon which is intended to convey some information.

A signal can be represented as a function of one or more independent variables.

$$\underset{\text{Dependent variable}}{x} \coloneqq f\,\underbrace{(t_1, t_2, t_3, \ldots, t_n)}_{\text{Independent variables}}$$

1.2 Continuous Time and Discrete Time

1.2.1 Continuous Time

A Continuous Time (CT) signal is defined as a signal $x(t)$ such that $x(t)$ has a well-defined value $\forall\ t$.

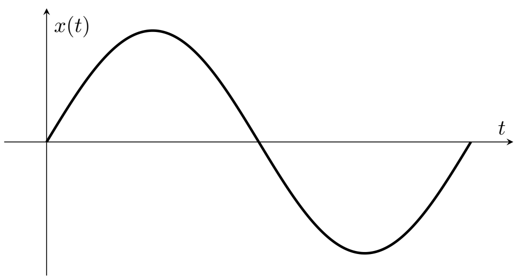


Figure 1.1: A CT signal

1.2.2 Discrete Time

A Discrete Time (DT) signal is defined as a signal x_n (or $x[n]$) which has a well-defined value for only certain values. Alternatively, if t is a *discrete variable* for a signal $x(t)$, then $x(t)$ is known as a DT signal.

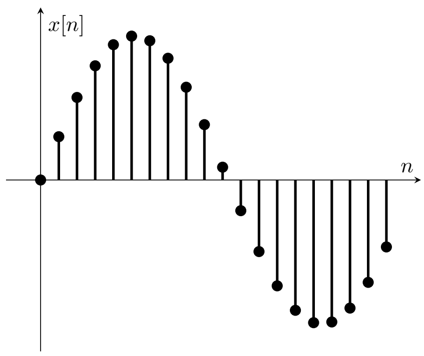


Figure 1.2: A DT signal

1.3 Classification of signals

1.3.1 Analog and Digital signals

A CT signal $x(t)$ is said to be *analog* if it is well-defined for any value in an interval (a, b).

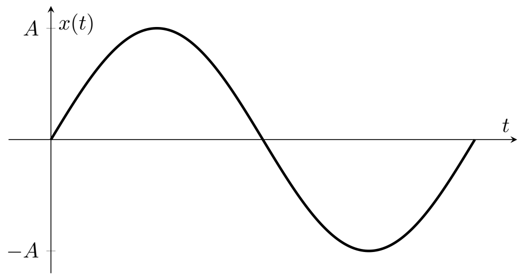


Figure 1.3: An analog signal

A CT signal $x(t)$ is said to be *digital* if it can only take a finite number of distinct values $\{x_1, x_2, x_3, \dots\}$.

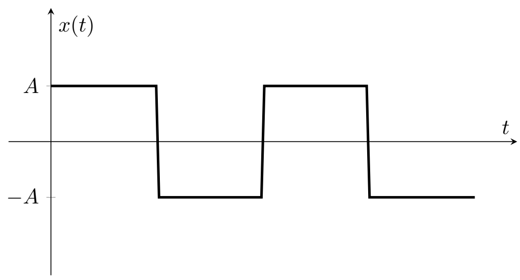


Figure 1.4: A digital signal

1.3.2 Real and Complex signals

A signal $x(t)$ is said to be *real* if $\forall\ t\ x(t) \in \mathbb{R}$.

A comlpex signal $x(t)$ is defined as $x(t) \coloneqq x_1(t) + jx_2(t)$, where $x_1(t)$ and $x_2(t)$ are real signals.

1.3.3 Deterministic and Random signals

Deterministic signals are the signals whose values are well-defined for any t. They can be modelled after a known function of t, such as a sinusoid.

Random or *non-determinisitc* signals cannot be determined for all t. They must be analyzed by statistical methods.

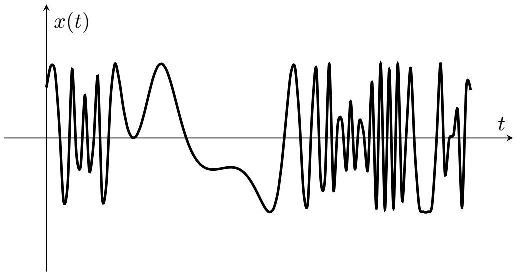


Figure 1.5: A random signal

1.3.4 Symmetric and Antisymmetric signals

A signal $x(t)$ or $x[n]$ is said to be *symmetric* (or *even*) iff $x(-t) = x(t)$ or $x[-n] = x[n]$.

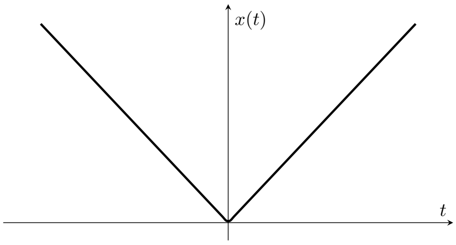


Figure 1.6: An even CT signal

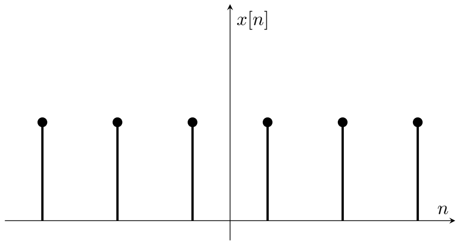


Figure 1.7: An even DT signal

A signal $x(t)$ or $x[n]$ is said to be *antisymmetric* (or *odd*) iff $x(-t) = -x(t)$ or $x[-n] = -x[n]$.

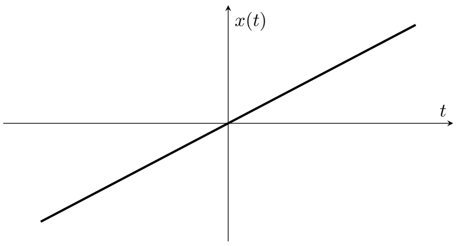


Figure 1.8: An odd CT signal

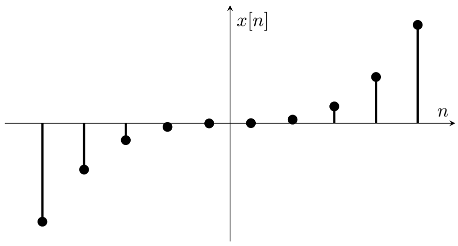


Figure 1.9: An odd DT signal

Any signal $x(t)$ can be expressed as a sum of its even function $x_e(t)$ and its odd function $x_o(t)$.

$$x(t) = x_e(t) + x_o(t) \tag{1.1}$$

$$x_e(t) \coloneqq \frac{1}{2}(x(t) + x(-t)) \tag{1.2}$$

$$x_o(t) \coloneqq \frac{1}{2}(x(t) - x(-t)) \tag{1.3}$$

1.3.5 Periodic and Non-periodic signals

A signal $x(t)$ is said to be *periodic* if $\exists\ T > 0\ \forall\ t\ :\ x(t+T) = x(t)$. If this condition is not followed, then the function is called *non-periodic*.

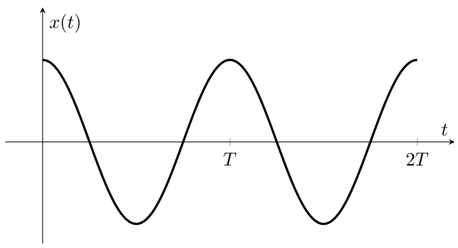


Figure 1.10: A periodic signal

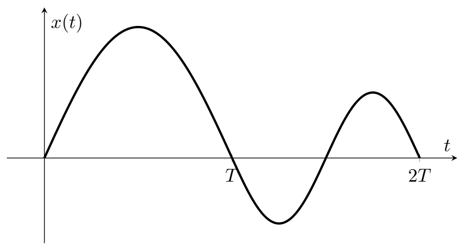


Figure 1.11: A non-periodic signal

1.4 Energy and Power

Consider $x(t)$ or $x[n]$ to be arbitrary time-varying signals.

1.4.1 Energy

Energy is denoted by E. The total energy of the signal x can be computed by the following equations:

$$\text{For a CT signal, } E \coloneqq \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t \tag{1.4}$$

$$\text{For a DT signal, } E \coloneqq \sum_{\text{all } n} |x[n]|^2 \tag{1.5}$$

1.4.2 Power

Power is denoted by P. The normalized (or average) power of the signal x can be computed by the following equations:

$$\text{For a CT signal, } P \coloneqq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, \mathrm{d}t \tag{1.6}$$

$$\text{For a DT signal, } P \coloneqq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{1.7}$$

1.5 Standard CT signals

1.5.1 Step signal

A step signal is a digital signal that changes its amplitude once.

Unit step signal

The unit step signal is defined as follows:

$$u(t) \coloneqq \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.8}$$

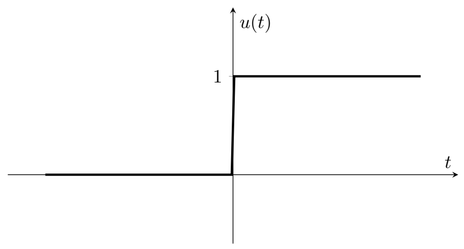


Figure 1.12: Unit step signal

Shifted unit step signal

The unit step signal shifted by time t_0 is defined as follows:

$$u(t - t_0) := \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases} \tag{1.9}$$

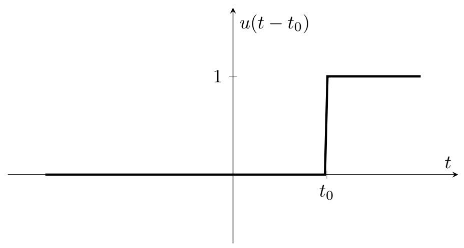


Figure 1.13: Shifted unit step signal

1.5.2 Impulse signal

An impulse is a short period of high amplitude. The impulse signal is defined as follows:

$$\delta(t) := \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{1.10}$$

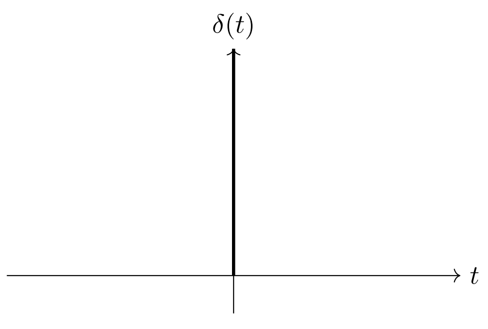


Figure 1.14: Impulse signal

Shifted impulse

The impulse signal shifted by time t_0 is defined as follows:

$$\delta(t-t_0) \coloneqq \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases} \tag{1.11}$$

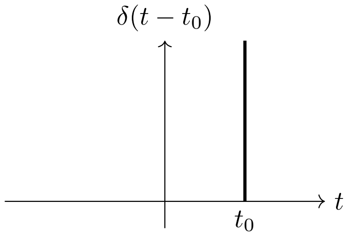


Figure 1.15: Shifted impulse signal

1.5.3 Complex exponential signal

The complex exponential signal is defined as follows:

$$x(t) \coloneqq e^{j\omega t} \tag{1.12}$$
$$= \cos\omega t + j\sin\omega t \tag{1.13}$$

Fundamental period, $T = \dfrac{2\pi}{\omega}$.

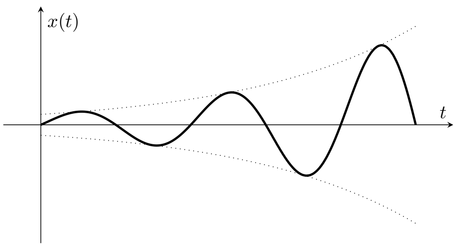


Figure 1.16: Exponentially increasing signal

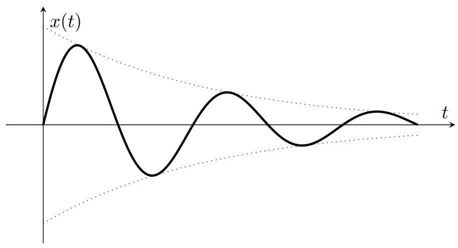


Figure 1.17: Exponentially decreasing signal

1.5.4 Ramp signal

The ramp signal $r(t)$ is defined as follows:

$$r(t) \coloneqq \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.14}$$

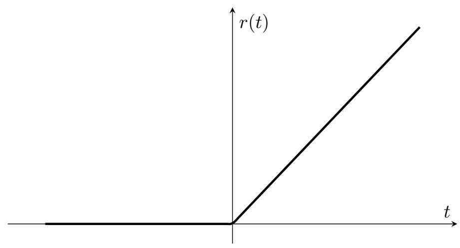


Figure 1.18: Ramp signal

1.5.5 Parabolic signal

The parabolic signal $x(t)$ is defined as follows:

$$x(t) \coloneqq \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.15}$$

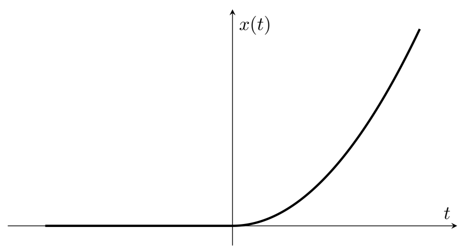


Figure 1.19: Parabolic signal

1.5.6 Rectangular signal

A rectangular signal $x(t)$ is defined for a period T as follows:

$$x(t) \coloneqq \begin{cases} 1, & t \in [-T/2, T/2] \\ 0, & \text{otherwise} \end{cases} \tag{1.16}$$

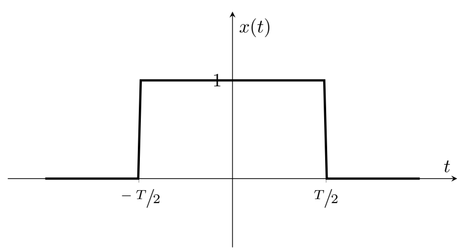


Figure 1.20: Rectangular signal

1.5.7 Triangular signal

A triangular signal $x(t)$ is defined for a period T as follows:

$$x(t) \coloneqq \begin{cases} 1 + {}^{2t}/_{T}, & t \in [-{}^{T}/_{2}, 0) \\ 1 - {}^{2t}/_{T}, & t \in [0, {}^{T}/_{2}] \\ 0, & \text{otherwise} \end{cases} \tag{1.17}$$

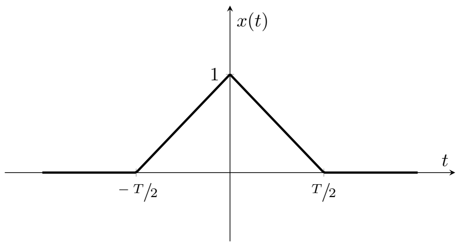


Figure 1.21: Triangular signal

1.5.8 Sinusoid

A sinusoidal signal $x(t)$ is given by the following equation:

$$x(t) \coloneqq \underbrace{A}_{\text{Amplitude}} \sin(\overbrace{\omega}^{\text{Angular frequency}} t + \underbrace{\varphi}_{\text{Initial phase}}) \tag{1.18}$$

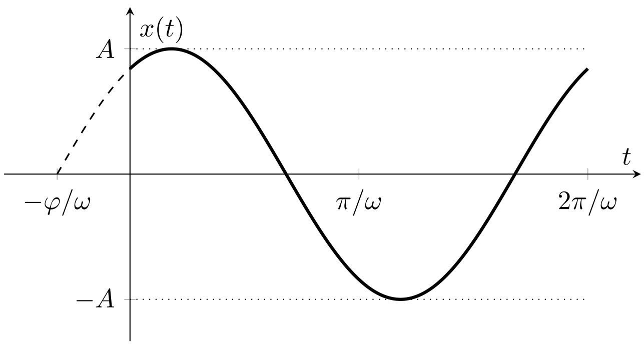


Figure 1.22: Sinusoidal signal

1.6 Basic signal operations

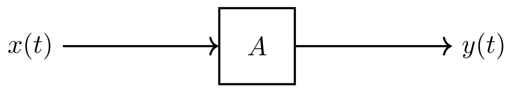
These are the fundamental operations that can be performed on signals.

Signals can be operated on using amplitude or time.

1.6.1 Gain

Given an input signal $x(t)$ and an output signal $y(t)$, the gain A is defined as,

$$\text{Gain, } A \coloneqq \frac{y(t)}{x(t)} \tag{1.19}$$



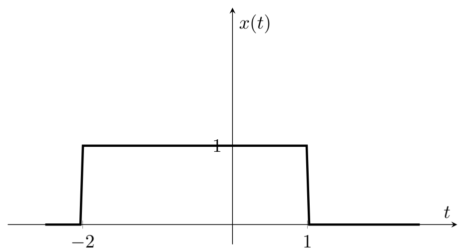
$$A > 1 \implies \text{Amplification}$$
$$0 < A < 1 \implies \text{Attenuation}$$

1.6.2 Amplitude operations

Amplitude scaling

Consider $x(t) := \begin{cases} 1, & t \in [-2, 1] \\ 0, & \text{otherwise} \end{cases}$

Let $y_1(t) := 2x(t) = \begin{cases} 2, & t \in [-2, 1] \\ 0, & \text{otherwise} \end{cases}$ and $y_2(t) := 0.5x(t) = \begin{cases} 0.5, & t \in [-2, 1] \\ 0, & \text{otherwise} \end{cases}$.



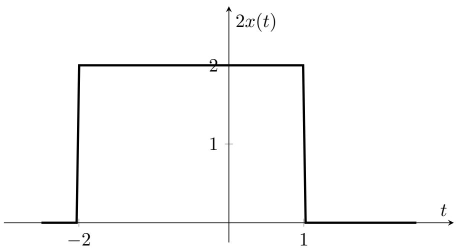


Figure 1.23: Amplitude scaling: Amplification

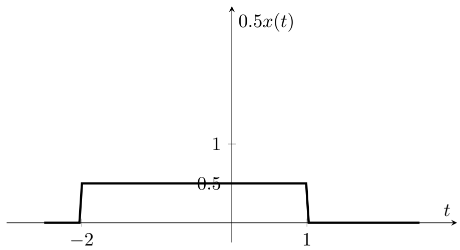


Figure 1.24: Amplitude scaling: Attenuation

Amplitude reversal

If $x(t) := \begin{cases} 1, & t \in [-2, 1] \\ 0, & \text{otherwise} \end{cases}$ then the amplitude-reversed signal $-x(t) = \begin{cases} -1, & t \in [-2, 1] \\ 0, & \text{otherwise} \end{cases}$.

Addition of signals

Consider $x_1(t) := \begin{cases} 1, & t \in [-1, 3] \\ 0, & \text{otherwise} \end{cases}$ and $x_2(t) := \begin{cases} 2, & t \in [0, 2] \\ 0, & \text{otherwise} \end{cases}$.

Their sum $y(t) := x_1(t) + x_2(t) = \begin{cases} 1, & t \in [-1, 0) \cup [2, 3] \\ 3, & t \in [0, 2) \\ 0, & \text{otherwise} \end{cases}$.

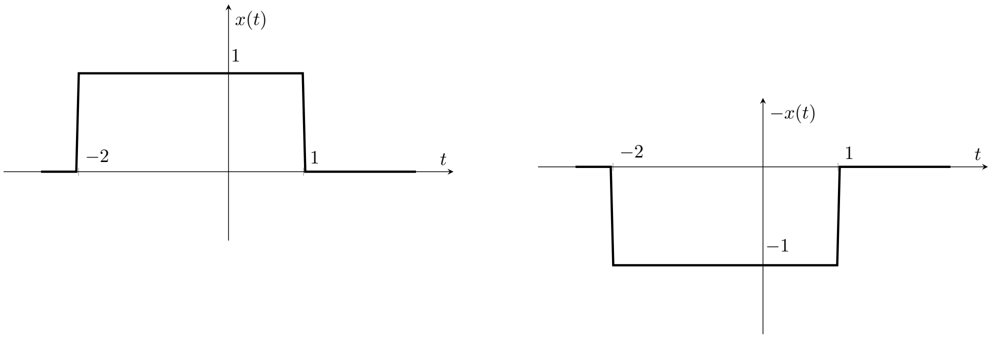


Figure 1.25: Amplitude reversal

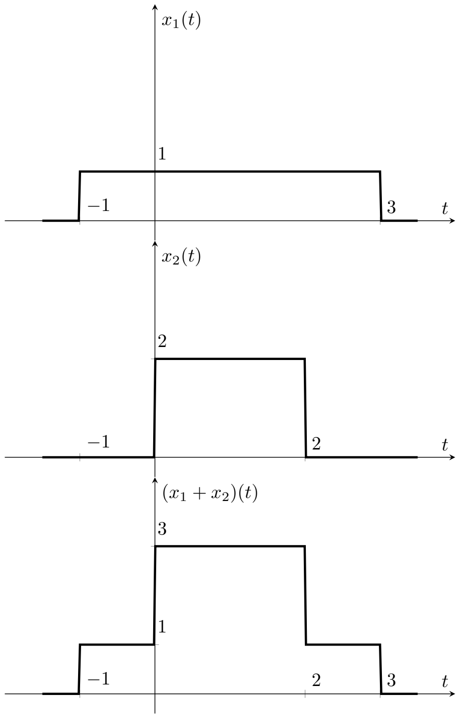


Figure 1.26: Addition of signals

Multiplication of signals

Consider $x_1(t) \coloneqq \begin{cases} 1, & t \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$ and $x_2(t) \coloneqq \begin{cases} 2, & t \in [0, 2] \\ 0, & \text{otherwise} \end{cases}$.

Their product $y(t) \coloneqq x_1(t) \cdot x_2(t) = \begin{cases} 2, & t \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$.

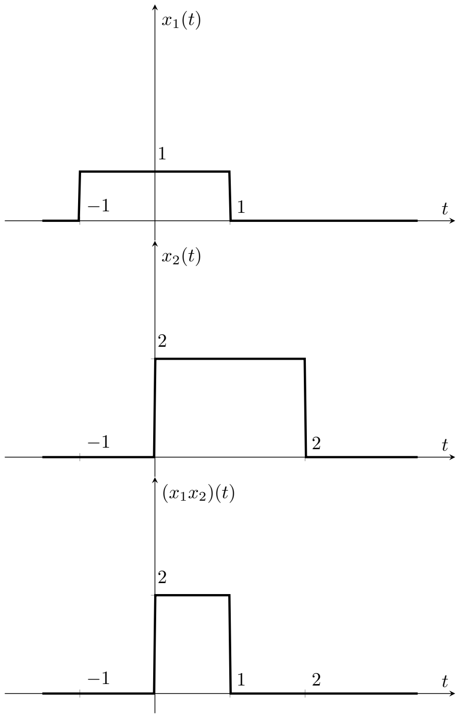


Figure 1.27: Multiplication of signals

Differentiation

The *derivative* of a signal $x(t)$ with respect to time at a given point in time describes the slope of the tangent to the signal at that point in time.

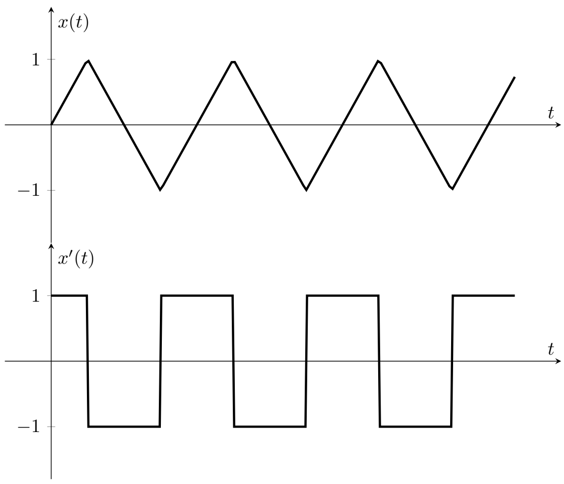


Figure 1.28: Differentiation

Integration

The *integral* of a signal $x(t)$ with respect to time describes the area under the plot of the signal.

1.6.3 Time operations

The operations on the independent variable time are also known as *time transformations*.

Time shift

Time shifting changes the position of the signal on the time axis, but not its shape.

$$x(t) \longrightarrow \underbrace{x(t-t_0)}_{\text{Delay by } t_0} \qquad \text{(Right shift)}$$

$$x(t) \longrightarrow \underbrace{x(t+t_0)}_{\text{Advance by } t_0} \qquad \text{(Left shift)}$$

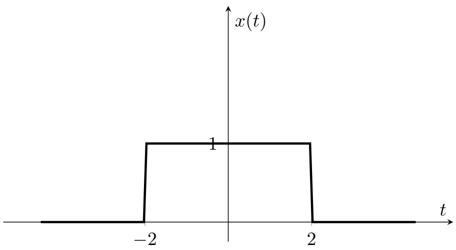
Time scaling

When we manipulate the time axis of a signal by multiplying a constant to the independent variable time t, the signal gets compressed or expanded depending on the value of the scaling factor α.

$$\therefore x(t) \longrightarrow x(\alpha t)$$

$$\alpha > 1 \implies \text{Compression}$$
$$0 < \alpha < 1 \implies \text{Expansion}$$

For example, consider $x(t) := \begin{cases} 1, & t \in [-2, 2] \\ 0, & \text{otherwise} \end{cases}$



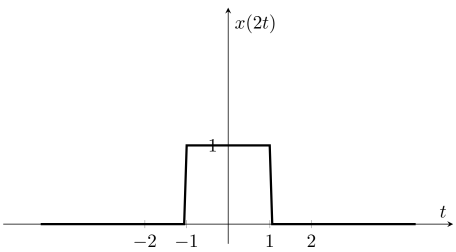


Figure 1.29: Time scaling: Compression

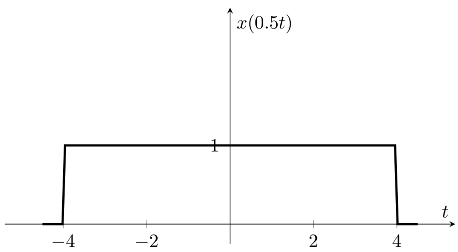


Figure 1.30: Time scaling: Expansion

Time reversal

Consider $x(t) \coloneqq \begin{cases} 1, & t \in [-2, 0) \\ 1 - t, & t \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$. The time-reversed signal $x(-t) = \begin{cases} 1 + t, & t \in [-1, 0] \\ 1, & t \in (0, 2] \\ 0, & \text{otherwise} \end{cases}$.

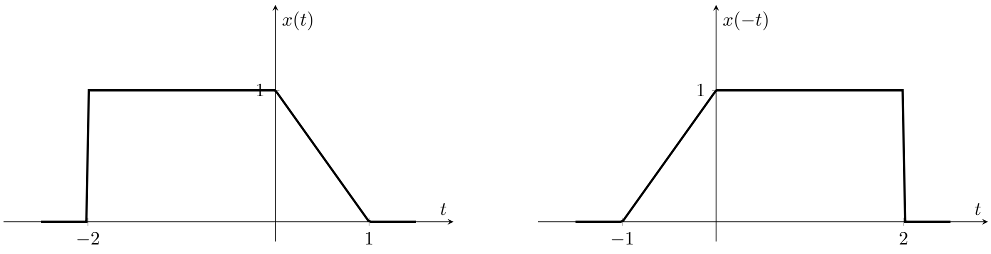


Figure 1.31: Time reversal

- Even signals are invariant under time reversal.
- Time reversal of odd signals causes amplitude reversal.

Chapter 2

Z transform

2.1 CT transformations

2.1.1 Fourier transform

A *Fourier transform* is used to convert a signal from time domain to frequency domain. It is a mathematical tool used for easier analysis of signals.

One application of Fourier transform is to convert the convolution operation to a multiplication operation between two signals.

$$\text{Convolution}\,(*) \underset{\text{Inverse Fourier transform}}{\overset{\text{Fourier transform}}{\rightleftharpoons}} \text{Multiplication}\,(\cdot)$$

A Fourier transform exists for Energy signals, Power signals and Impulse-related signals *i.e.* signals that are *absolutely integrable*. Fourier transform does not exist for signals that are not absolutely integrable, such as Neither Energy nor Power (NENP) signals.

The Fourier transform $X(j\omega)$ of a signal $x(t)$ is defined as follows:

$$X(j\omega) = \mathcal{F}\{x(t)\} \coloneqq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t \tag{2.1}$$

The inverse Fourier transform for $X(j\omega)$ is given by:

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} \coloneqq \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}\,\mathrm{d}\omega \tag{2.2}$$

2.1.2 Laplace transform

A *Laplace transform* is a generalized Fourier transform. The transformed signal $F(s)$ of the signal $f(t)$ is given as,

$$X(s) = \mathcal{L}\{x(t)\} \coloneqq \int_{-\infty}^{\infty} f(t)e^{-st}\,\mathrm{d}t \qquad \text{(Bilateral Laplace transform)} \tag{2.3}$$

$$X(s) = \mathcal{L}\{x(t)\} \coloneqq \int_{0}^{\infty} f(t)e^{-st}\,\mathrm{d}t \qquad \text{(Unilateral Laplace transform)} \tag{2.4}$$

Here, s is a complex variable given as,

$$\underset{\text{Complex frequency}}{s} := \underset{\text{Damping factor}}{\sigma} + j\,\overset{\text{Angular frequency}}{\omega}$$

The inverse Laplace transform is given by,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi}\int_{-s}^{s} F(s)e^{st}\,\mathrm{d}s \tag{2.5}$$

A Laplace transform exists for Energy signals, Power signals, as well as certain NENP signals.

2.1.3 Relation between Laplace and Fourier transforms

Consider a Laplace transform $F(s)$ as follows:

$$F(s) := \int_{-\infty}^{\infty} f(t)e^{st}\,\mathrm{d}t$$

Substituting $s := \sigma + j\omega$,

$$\begin{aligned}
\therefore F(s) &= \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t)e^{-\sigma t-j\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \underbrace{f(t)e^{-\sigma t}}_{:=x(t)}\,e^{-j\omega t}\,\mathrm{d}t \\
\therefore F(s) &= \mathcal{F}\{x(t)\} = \mathcal{F}\big\{f(t)e^{-\sigma t}\big\} \implies \boxed{\mathcal{L}\{f(t)\} = \mathcal{F}\big\{f(t)e^{-\sigma t}\big\}}
\end{aligned} \tag{2.6}$$

2.2 Z-transform

A *Z-transform* is an analogous transformation to Laplace transforms but for DT signals. Just like Laplace transform, a Z-transform exists for Energy signals, Power signals and some NENP signals. The Z-transformed signal $x[z]$ of a DT signal $x[n]$ is given as follows:

$$X[z] = \mathcal{Z}\{x[n]\} := \lim_{N\to\infty}\sum_{n=-N}^{N} x[n]z^{-n} \qquad \text{(Bidirectional Z-transform)} \tag{2.7}$$

$$X[z] = \mathcal{Z}\{x[n]\} := \lim_{N\to\infty}\sum_{n=0}^{N} x[n]z^{-n} \qquad \text{(Unidirectional Z-transform)} \tag{2.8}$$

Where z is a complex number as follows:

$$\underset{\text{Magnitude}}{r}\; e^{\overset{}{\underset{\text{Complex argument}}{j\omega}}}$$

A signal can be transformed back using an *inverse Z-transform.*

$$x[n] \underset{\text{Inverse Z-transform}}{\overset{\text{Z-transform}}{\rightleftharpoons}} X[z]$$

2.2.1 Region of Convergence

The Region of Convergence (ROC) is the region in the z-plane where the Z-transform exists, convergent and finite.

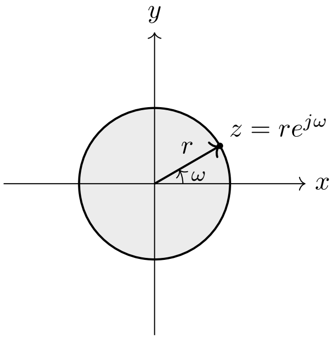
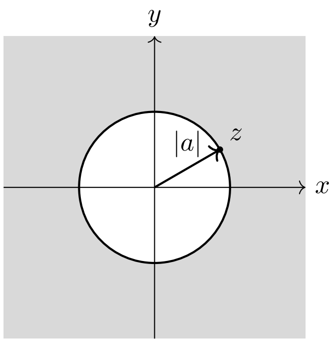


Figure 2.1: Region of Convergence (ROC)

Q.1. Consider the signal $x[n] \coloneqq a^n u[n]$. What is its region of convergence?

$$
\begin{aligned}
X[z] &\coloneqq \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\
&= \sum_{n=-\infty}^{0} a^n \cdot 0 \cdot z^{-n} + \sum_{n=0}^{\infty} a^n \cdot 1 \cdot z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \\
&= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots \\
&= \boxed{\frac{z}{z-a}}
\end{aligned}
$$

However, the above geometric series is convergent if and only if $\left|\frac{a}{z}\right| < 1 \implies |z| > |a|$. Therefore, the region of convergence is $r > |a|$.



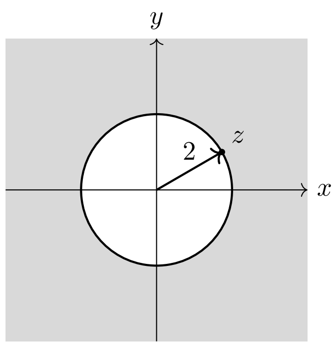
$|z| > |a|$

Properties of ROC

- ROC of Z-transform is the region surrounding a ring or disc in the z-plane centered at the origin.
- ROC of a Z-transform does not contain any poles ($\div\, 0$).
- ROC of an Linear Time-Invariant (LTI) stable system has a unit circle.
- For $\delta[n]$, ROC is the entire z-plane.
- The ROC of an infinite duration causal (right-sided) sequence is the exterior of a circle ($|z| > a$ for some a).
- The ROC of an infinite duration anticausal (left-sided) sequence is the interior of a circle ($|z| > a$ for some a).
- The ROC of an infinite duration two-sided DT sequence is an annulus or ring ($a < |z| < b$ for some a, b)).
- The ROC of a finite duration causal sequence is the entire z-plane except for $z = 0$.
- The ROC of a finite duration anticausal sequence is the entire z-plane except for $z = \infty$.
- The ROC of a finite duration two-sided sequence is the entire z-plane except for $z = 0$ and $z = \infty$.
- The ROC of the sum of two signals is the intersection of their ROCs.

Q.2. Find the ROC of the signal $x[n] \coloneqq 2^n u[n]$.

$\because$ $2^n u[n]$ is an infinite duration causal sequence, its region of convergence the exterior of the circle of radius 2 centered at origin, *i.e.* $|z| > 2$.



$|z| > 2$

2.2.2 Properties of Z-transform

Linearity Consider the DT signals $x_1[n]$ and $x_2[n]$ as follows:

$$
\begin{aligned}
x_1[n] &\rightleftharpoons X_1[z] & \text{ROC} \coloneqq R_1 \\
x_2[n] &\rightleftharpoons X_2[z] & \text{ROC} \coloneqq R_2 \\
\text{Scaling by } \alpha : \alpha x_1[n] &\rightleftharpoons \alpha X_1[z] \\
\text{Scaling by } \beta : \beta x_2[n] &\rightleftharpoons \beta X_2[z]
\end{aligned}
$$

$$\therefore \alpha x_1[n] + \beta x_2[n] \rightleftharpoons \alpha X_1[z] + \beta X_2[z],\ \text{ROC} \supseteq R_1 \cap R_2 \tag{2.9}$$

Time Scaling Consider a DT signal $x[n]$ as follows:

$$\begin{aligned} x[n] &\rightleftharpoons X[z] && \text{ROC} \coloneqq R \\ \therefore x\left[\frac{n}{m}\right] &\rightleftharpoons X[z^m] && \text{ROC} \coloneqq R^{1/m} \end{aligned} \tag{2.10}$$

Time Shifting Consider a DT signal $x[n]$ as follows:

$$\begin{aligned} x[n] &\rightleftharpoons X[z] && \text{ROC} \coloneqq R \\ \therefore x[n-n_0] &\rightleftharpoons z^{-n_0}X[z] && \text{ROC} \coloneqq R \end{aligned} \tag{2.11}$$

Time Reversal Consider a DT signal $x[n]$ as follows:

$$\begin{aligned} x[n] &\rightleftharpoons X[z] && \text{ROC} \coloneqq R \\ \therefore x[-n] &\rightleftharpoons X[z^{-1}] && \text{ROC} \coloneqq {}^{1}/_{R} \end{aligned} \tag{2.12}$$

Amplitude Scaling Consider a DT signal $x[n]$ as follows:

$$\begin{aligned} x[n] &\rightleftharpoons X[z] && \text{ROC} \coloneqq R \\ \therefore a^n x[n] &\rightleftharpoons X\left[\frac{z}{a}\right] && \text{ROC} \coloneqq |a|R \end{aligned} \tag{2.13}$$

Conjugation Consider a DT signal $x[n]$ as follows:

$$\begin{aligned} x[n] &\rightleftharpoons X[z] && \text{ROC} \coloneqq R \\ \therefore x^*[n] &\rightleftharpoons X^*[z^*] && \text{ROC} \coloneqq R \end{aligned} \tag{2.14}$$

Accumulation Consider a DT signal $x[n]$ as follows:

$$\begin{aligned} x[n] &\rightleftharpoons X[z] && \text{ROC} \coloneqq R \\ \therefore \sum_{k=-\infty}^{n} x[k] &\rightleftharpoons \frac{X[z]}{1-z^{-1}} && \text{ROC} \supseteq R \cap (|z| > 1) \end{aligned} \tag{2.15}$$

Acronyms

CT Continuous Time 4–7

DT Discrete Time 4–7, 18, 20, 21

LTI Linear Time-Invariant 20

NENP Neither Energy nor Power 17, 18

ROC Region of Convergence 19–21

RRG Rupak R. Gupta 1

$$\boxed{\begin{bmatrix} \mathbb{R} & \mathfrak{R} \\ g & \infty \end{bmatrix}}$$

The End.