Arrows or Arrays?

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Contents

Chapter 1

Linear Algebra

1.1 Matrices

A matrix is a rectangular array of expressions, used to represent a mathematical property or object.

Order of a matrix is given by $m \times n$, where m and n represent the number of rows and the number of columns respectively.

The set of all matrices of order $m \times n$ containing elements of a scalar field $\mathbb{F}$ is represented by $\mathcal{M}_{m \times n}(\mathbb{F})$.

For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

1.1.1 Matrix transformations

Transpose

The transpose $A^\top$ of a matrix A is obtained by switching its rows and columns.

$$A^\top \coloneqq [a_{ji}] \; \forall \; a_{ij} \in A \tag{1.1}$$

If $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, then $A^\top \in \mathcal{M}_{n \times m}(\mathbb{F})$.

Adjugate

Inverse

The inverse A^{-1} of a matrix A is defined such that their product results in the identity matrix I, which is discussed further in §1.1.2.

$$AA^{-1} = A^{-1}A = I \tag{1.2}$$

The inverse can be obtained from the adjugate using the following definition,

$$A^{-1} \coloneqq \frac{\operatorname{adj} A}{\det(A)} \tag{1.3}$$

The determinant will be discussed further in §1.1.3.

1.1.2 Properties of matrices

Square matrices

If the number of rows of a matrix is the same as the number of its columns, then the matrix is said to be a square matrix. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Symmetric matrices

Skew-symmetric matrices

Null matrices

Orthogonal matrices

Idempotent matrices

Nilpotent matrices

Diagonal matrices

Identity matrices

1.1.3 Operations on square matrices

Trace

The *trace* of a square matrix is defined to be the sum of the elements on its leading diagonal.

$$\operatorname{tr}\left(\begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}\right) \coloneqq m_{11} + m_{22} + m_{33} + \cdots + m_{nn} = \sum_{i=1}^{n} m_{ii} \tag{1.4}$$

Determinant

Elementary Row Operations

Row Echelon form of a matrix

1.1.4 Linear Dependence of Vectors

1.1.5 System of Linear Equations

1.1.6 Similarity of two matrices

1.2 Vector Spaces

A set V with operations of addition and scalar multiplication over a field $\mathbb{F}$ defined on it is called a vector space if it satisfies the following axioms:

Closure

1. $\forall\, \mathbf{u}, \boldsymbol{v} \in V\ (\mathbf{u} + \boldsymbol{v} \in V)$ (Closure under vector addition)
2. $\forall\, \boldsymbol{v} \in V\ \forall\, \lambda \in \mathbb{F}\ (\lambda \boldsymbol{v} \in V)$ (Closure under scalar multiplication)

Vector Addition

3. $\forall\, \mathbf{u}, \boldsymbol{v} \in V\ (\mathbf{u} + \boldsymbol{v} = \boldsymbol{v} + \mathbf{u})$ (Commutativity)
4. $\forall\, \mathbf{u}, \boldsymbol{v}, \boldsymbol{w} \in V\ (\mathbf{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\mathbf{u} + \boldsymbol{v}) + \boldsymbol{w})$ (Associativity)
5. $\exists\, \mathbf{0} \in V\ \forall\, \boldsymbol{v} \in V\ (\boldsymbol{v} + \mathbf{0} = \boldsymbol{v})$ (Existence of neutral element)
6. $\forall\, \boldsymbol{v} \in V\ \exists\, (-\boldsymbol{v}) \in V\ (\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0})$ (Existence of additive inverse)

Scalar Multiplication

7. $\forall\, \mathbf{u}, \boldsymbol{v} \in V\ \forall\, \lambda \in \mathbb{F}\ (\lambda(\mathbf{u} + \boldsymbol{v}) = \lambda \mathbf{u} + \lambda \boldsymbol{v})$ (Distributivity over vector addition)
8. $\forall\, \boldsymbol{v} \in V\ \forall\, \lambda, \mu \in \mathbb{F}\ ((\lambda + \mu)\boldsymbol{v} = \lambda \boldsymbol{v} + \mu \boldsymbol{v})$ (Distributivity over field addition)
9. $\forall\, \boldsymbol{v} \in V\ \forall\, \lambda, \mu \in \mathbb{F}\ ((\lambda\mu)\boldsymbol{v} = \lambda(\mu \boldsymbol{v}))$ (Associativity)
10. $\exists\, 1 \in \mathbb{F}\ \forall\, \boldsymbol{v} \in V\ (1\boldsymbol{v} = \boldsymbol{v})$ (Existence of neutral element)

1.2.1 Vector Subspaces

A set W is said to be a *subspace* of a vector space V over a field $\mathbb{F}$, if itself is also a vector space. Since V is given to be a vector space, W needs to satisfy only the following axioms:

1. $\mathbf{0} \in W$ (Existence of additive identity)
2. $\forall\, \mathbf{u}, \boldsymbol{v} \in W\ (\mathbf{u} + \boldsymbol{v} \in W)$ (Closure under addition)
3. $\forall\, \boldsymbol{v} \in W\ \forall\, \lambda \in \mathbb{F}\ (\lambda \boldsymbol{v} \in W)$ (Closure under scalar multiplication)

1.2.2 Span

The span of a set of vectors is defined as the set of all linear combinations of those vectors.

$$\operatorname{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n) \coloneqq \left\{ \boldsymbol{w} \;\middle|\; \boldsymbol{w} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i \;\forall\, \lambda_i \in \mathbb{R} \right\} \tag{1.5}$$

1.2.3 Basis

A subset S of a vector space V is said to form a basis for V if $\operatorname{span}(S) = V$. This is verified if the following is true:

1. S is linearly independent.
2. $\dim(S) = \dim(V)$

Therefore, $\mathbf{0}$ can never exist in a basis.

Basis vectors are usually denoted by $\boldsymbol{e}_i$, where i refers to the i^{th} basis vector.

1.3 Linear Transformations

A transformation T is a mapping from a vector space V to a vector space W over a field $\mathbb{F}$. A transformation is said to be linear if it follows the following axioms:

1. $T(\mathbf{0}_V) = \mathbf{0}_W$ (Origin invariance)
2. $\forall\, \mathbf{u}, \boldsymbol{v} \in V\ (T(\mathbf{u} + \boldsymbol{v}) = T(\mathbf{u}) + T(\boldsymbol{v}))$ (Linearity over vector addition)
3. $\forall\, \boldsymbol{v} \in V\ \forall\, \lambda \in \mathbb{F}\ (T(\lambda \boldsymbol{v}) = \lambda \cdot T(\boldsymbol{v}))$ (Linearity over scalar multiplication)

1.3.1 Column space

The column space (or image) of a linear transformation $T : V \to W$ is defined as the span of all transformed vectors in the codomain W.

$$\operatorname{im}(T) \coloneqq \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in V\} \tag{1.6}$$

The *rank* of a transformation is defined as the dimension of its column space.

$$\operatorname{rank}(T) \coloneqq \dim(\operatorname{im}(T)) \tag{1.7}$$

1.3.2 Null space

The null space (or kernel) of a linear transformation $T : V \to W$ is defined as the set of all vectors in the domain V that map to the additive identity in the codomain W.

$$\ker(T) \coloneqq \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \mathbf{0}_W\} \tag{1.8}$$

The *nullity* of a transformation is defined as the dimension of its null space.

$$\operatorname{nullity}(T) \coloneqq \dim(\ker(T)) \tag{1.9}$$

1.3.3 Rank-Nullity Theorem

The rank-nullity theorem states that for a linear transformation T, the sum of the dimensions of its column space and its null space equals the dimension of the domain.

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V) \tag{1.10}$$

1.3.4 Inner Product

The inner product is defined as a mapping from a pair of vectors belonging to a vector space to a scalar.

For vectors $\mathbf{u} \coloneqq \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\boldsymbol{v} \coloneqq \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ their inner product is given by,

$$\langle \mathbf{u}, \boldsymbol{v} \rangle \coloneqq u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{1.11}$$

Inner Product Space

A vector space V over a field $\mathbb{F}$ is said to be an inner product space if an inner product defined on it obeys the following axioms:

1. $\forall\, \mathbf{u}, \boldsymbol{v} \in V \; (\langle \mathbf{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \mathbf{u} \rangle)$
2. $\forall\, \mathbf{u}, \boldsymbol{v}, \boldsymbol{w} \in V \; (\langle \mathbf{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \mathbf{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle)$
3. $\forall\, \mathbf{u}, \boldsymbol{v} \in V \,\forall\, \lambda \in \mathbb{F} \; (\langle \lambda \mathbf{u}, \boldsymbol{v} \rangle = \lambda \langle \mathbf{u}, \boldsymbol{v} \rangle)$
4. $\forall\, \boldsymbol{v} \in V \; (\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geqslant 0)$

Corollary

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \iff \boldsymbol{v} = \mathbf{0} \tag{1.12}$$

Norm

The norm of a vector $\boldsymbol{v} \coloneqq \langle v_1, v_2, \ldots, v_n \rangle$ in a vector space V equipped with an inner product is given by,

$$\begin{aligned} \|\boldsymbol{v}\| &\coloneqq \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} && (1.13) \\ &= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} && (1.14) \end{aligned}$$

The norm also represents the *length* of a vector.

Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality states that the inner product of any two vectors $\mathbf{u} \coloneqq \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} \coloneqq \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is always less than or equal to the product of their norms.

$$\langle \mathbf{u}, \mathbf{v} \rangle \leqslant \|\mathbf{u}\| \|\mathbf{v}\| \tag{1.15}$$

$$\therefore u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \leqslant \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\implies \sum_{i=1}^{n} u_i v_i \leqslant \sqrt{\sum_{i=1}^{n} u_i^2} \sqrt{\sum_{i=1}^{n} v_i^2} \tag{1.16}$$

Cosine Similarity

The angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by,

$$\theta \coloneqq \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}\right) \implies \cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \tag{1.17}$$

1.3.5 Orthogonal Transformation

A linear transformation $T : V \to W$ is said to be an *orthogonal transformation* if the following axioms hold:

1. $\forall \mathbf{v} \in V \, (\|T(\mathbf{v})\| = \|\mathbf{v}\|)$ (Length invariance)
2. $\forall \mathbf{u}, \mathbf{v} \in V \left(\arccos\left(\frac{\langle T(\mathbf{u}), T(\mathbf{v}) \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}\right) = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)\right)$ (Angle invariance)

The resultant of these axioms can be given by the following relation:

$$\langle T(\mathbf{u}), T(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \tag{1.18}$$

1.3.6 Eigenvalues and Eigenvectors

Diagonalization

Cayley-Hamilton Theorem

Function of a square matrix

Chapter 2

Numerical Linear Algebra

2.1 Gaussian Elimination

Chapter 3

Vector Algebra

3.1 Definition

A vector is a mathematical entity that has a magnitude and a direction. A vector can be represented using an array of n numbers, where n is called the *dimension* of the vector.

3.2 Vector Operations

3.2.1 Vector Addition

Parallelogram law

$$\|\mathbf{u}+\mathbf{v}\| \coloneqq \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u}\cdot\mathbf{v})} \tag{3.1}$$

3.2.2 Vector Products

Scalar Product

If vectors $\mathbf{u} \coloneqq \langle u_1, u_2, \ldots, u_n\rangle$ and $\mathbf{v} \coloneqq \langle v_1, v_2, \ldots, v_n\rangle \in \mathbb{R}^n$, then their scalar product (or dot product) is defined as,

$$\mathbf{u}\cdot\mathbf{v} \coloneqq u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{3.2}$$

If θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then the scalar product has the following polar definition:

$$\mathbf{u}\cdot\mathbf{v} \coloneqq \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{3.3}$$

Vector Product

Consider vectors $\mathbf{u} \coloneqq \langle u_x, u_y, u_z\rangle$ and $\mathbf{v} \coloneqq \langle v_x, v_y, v_z\rangle \in \mathbb{R}^3$. Their vector product (or cross product) is defined as,

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &\coloneqq \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}\\ &= \langle u_yv_z - u_zv_y, u_zv_x - u_xv_z, u_xv_y - u_yv_x\rangle\end{aligned} \tag{3.4}$$

If θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then the vector product has the following polar definition:

$$\mathbf{u} \times \mathbf{v} \coloneqq (\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta)\hat{\mathbf{n}} \tag{3.5}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plane formed by the vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$, $\mathbf{v}$ and $\hat{\mathbf{n}}$ form a right-handed system.

Scalar Triple Product

The scalar triple product (or box product) of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is given by

$$\begin{aligned} \det\begin{pmatrix}\mathbf{u} & \mathbf{v} & \mathbf{w}\end{pmatrix} &\coloneqq \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} && (3.6) \\ &= \begin{vmatrix} \mathbf{u}\cdot\hat{\mathbf{i}} & \mathbf{u}\cdot\hat{\mathbf{j}} & \mathbf{u}\cdot\hat{\mathbf{k}} \\ \mathbf{v}\cdot\hat{\mathbf{i}} & \mathbf{v}\cdot\hat{\mathbf{j}} & \mathbf{v}\cdot\hat{\mathbf{k}} \\ \mathbf{w}\cdot\hat{\mathbf{i}} & \mathbf{w}\cdot\hat{\mathbf{j}} & \mathbf{w}\cdot\hat{\mathbf{k}} \end{vmatrix} && (3.7) \end{aligned}$$

Vector Triple Product

The vector triple product of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is given by

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &\coloneqq (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} && (3.8) \\ (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &\coloneqq (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} && (3.9) \end{aligned}$$

3.2.3 Projection

The vector projection of a vector $\mathbf{u}$ on a vector $\mathbf{v}$ is defined as

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} \coloneqq (\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} = \left(\mathbf{u}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{3.10}$$

The scalar projection is simply the norm of the above expression.

$$\|\mathrm{proj}_{\mathbf{v}}\mathbf{u}\| \coloneqq \mathbf{u}\cdot\hat{\mathbf{v}} = \mathbf{u}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{3.11}$$

3.2.4 Lagrange's Iidentity

Lagrange's identity states that,

$$(\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \tag{3.12}$$

3.3 Three Dimensional Geometry

3.3.1 Coplanarity

The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathbb{R}^3$ are said to be coplanar iff,

$$\det\begin{pmatrix}\boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w}\end{pmatrix} = 0$$
$$\text{or,} \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$$

3.3.2 Basis

The basis vectors of 3D space are defined as $\hat{\imath}$ for the x axis, $\hat{\jmath}$ for the y axis and $\hat{k}$ for the z axis.

They are related to each other as follows:

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \iff \hat{\jmath} \times \hat{\imath} = -\hat{k}$$
$$\hat{\jmath} \times \hat{k} = \hat{\imath} \iff \hat{k} \times \hat{\jmath} = -\hat{\imath}$$
$$\hat{k} \times \hat{\imath} = \hat{\jmath} \iff \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

A vector $\boldsymbol{v} \in \mathbb{R}^3$ with the components v_x, v_y and v_z can be denoted as $\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$, $v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ or $\langle v_x, v_y, v_z \rangle$.

3.3.3 Directional Cosines and Ratios

Consider a vector $\boldsymbol{v} \coloneqq \langle v_x, v_y, v_z \rangle \in \mathbb{R}^3$.

The normalized vector $\hat{\boldsymbol{v}}$ is given by,

$$\begin{aligned} \hat{\boldsymbol{v}} &\coloneqq \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \\ &= \left\langle \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}}, \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}}, \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \right\rangle \end{aligned} \tag{3.13}$$

The directional cosines l, m and n of $\boldsymbol{v}$ are given by,

$$l = \cos\alpha \coloneqq \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \tag{3.14}$$

$$m = \cos\beta \coloneqq \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \tag{3.15}$$

$$n = \cos\gamma \coloneqq \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \tag{3.16}$$

Therefore, any vector parallel to $\boldsymbol{v}$ is proportional to the vector $\langle l, m, n \rangle$.

$$\boldsymbol{u} \parallel \boldsymbol{v} \implies \boldsymbol{u} \propto \langle l, m, n \rangle \tag{3.17}$$

Here, $\mathbf{u}$ can be defined to be the vector $\langle a, b, c \rangle$, where a, b, c are known as directional ratios of $\mathbf{v}$.

Corollaries

$$l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{3.18}$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 \tag{3.19}$$

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} \tag{3.20}$$

3.3.4 Distance between two points

The distance between two points denoted by the position vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as $\|\mathbf{v} - \mathbf{u}\|$.

Section formula

A point $\mathbf{r}$ dividing the segment between vectors $\mathbf{u}$ and $\mathbf{v}$ in the ratio of $m_1 : m_2$ is given by

$$\mathbf{r} \coloneqq \frac{m_2\mathbf{u} + m_1\mathbf{v}}{m_1 + m_2} \text{ for internal division} \tag{3.21}$$

$$\mathbf{r} \coloneqq \frac{m_2\mathbf{u} - m_1\mathbf{v}}{m_1 - m_2} \text{ for external division} \tag{3.22}$$

3.3.5 Lines

Point-slope form

Let $\mathbf{a} \coloneqq \langle x_1, y_1, z_1 \rangle$ be a point on a line in the direction of the vector $\mathbf{b} \coloneqq \langle a, b, c \rangle$. A point $\mathbf{r} \coloneqq \langle x, y, z \rangle$ on the line is given by,

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R} \tag{3.23}$$

$$\therefore \mathbf{r} - \mathbf{a} = \lambda\mathbf{b}$$

$$\therefore \langle x - x_1, y - y_1, z - z_1 \rangle = \langle \lambda a, \lambda b, \lambda c \rangle$$

$$\implies x - x_1 = \lambda a,\ y - y_1 = \lambda b,\ z - z_1 = \lambda c$$

$$\therefore \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda \tag{3.24}$$

Eq. 3.24 is known as the *symmetric form* of the given line.

Two-point form

Let $\mathbf{a} \coloneqq \langle x_1, y_1, z_1 \rangle$ and $\mathbf{b} \coloneqq \langle x_2, y_2, z_2 \rangle$ be two points on a line. A point $\mathbf{r} \coloneqq \langle x, y, z \rangle$ on the line is given by,

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}),\ \lambda \in \mathbb{R} \tag{3.25}$$

$$\therefore \mathbf{r} - \mathbf{a} = \lambda(\mathbf{b} - \mathbf{a})$$

$$\therefore \langle x - x_1, y - y_1, z - z_1 \rangle = \langle \lambda(x_2 - x_1), \lambda(y_2 - y_1), \lambda(z_2 - z_1) \rangle$$

$$\implies x - x_1 = \lambda(x_2 - x_1),\ y - y_1 = \lambda(y_2 - y_1),\ z - z_1 = \lambda(z_2 - z_1)$$
$$\therefore \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda \tag{3.26}$$

Eq. 3.26 is known as the *symmetric two-point form* of the given line.

Unsymmetrical form

A line defined by the intersection of the planes given by $a_1x+b_1y+c_1z = d_1$ and $a_2x+b_2y+c_2z = d_2$ is given by,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{3.27}$$

This is known as the *unsymmetrical form* of the given line.

Angle between two lines

For the lines in the direction of the vectors $\mathbf{b}_1 \propto \langle l_1, m_1, n_1 \rangle$ and $\mathbf{b}_2 \propto \langle l_2, m_2, n_2 \rangle$, the angle (θ) between them is given by,

$$\theta = \arccos\left(\hat{\mathbf{b}}_1 \cdot \hat{\mathbf{b}}_2\right) = \arccos\left(\frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{\|\mathbf{b}_1\|\|\mathbf{b}_2\|}\right) = \arccos\left(l_1 l_2 + m_1 m_2 + n_1 n_2\right) \tag{3.28}$$

Distance between two lines

The distance between the parallel lines $L_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $L_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is given by,

$$\operatorname{dist}(L_1, L_2) = \frac{\|\mathbf{b} \times (\mathbf{a}_2 - \mathbf{a}_1)\|}{\|\mathbf{b}\|} \tag{3.29}$$

The distance between the skew lines $L_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $L_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ is given by,

$$\operatorname{dist}(L_1, L_2) = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)}{\|\mathbf{b}_1 \times \mathbf{b}_2\|}\right| \tag{3.30}$$

3.3.6 Planes

Point-normal form

Consider a plane containing a point $\mathbf{a} \coloneqq \langle x_1, y_1, z_1 \rangle$ and having a normal vector $\mathbf{n} \coloneqq \langle a, b, c \rangle$. A point $\mathbf{r} \coloneqq \langle x, y, z \rangle$ on the plane is given by,

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \qquad \text{or} \qquad \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \tag{3.31}$$
$$\therefore \langle x, y, z \rangle \cdot \langle a, b, c \rangle = \langle x_1, y_1, z_1 \rangle \cdot \langle a, b, c \rangle$$
$$\therefore ax + by + cz = ax_1 + by_1 + cz_1$$

Let $d \coloneqq ax_1 + by_1 + cz_1$.

$$\implies ax + by + cz = d \tag{3.32}$$

This is known as the *Cartesian equation* of the plane.

Two intersecting lines

Consider two intersecting lines given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{p}$ and $\mathbf{r} = \mathbf{a} + \mu\mathbf{q}$.

The normal vector $\mathbf{n}$ to the plane is given by $\mathbf{n} \coloneqq \mathbf{p} \times \mathbf{q}$. From eq. (3.31),

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{p} \times \mathbf{q}) = 0 \implies \det\begin{pmatrix} \mathbf{r} - \mathbf{a} & \mathbf{p} & \mathbf{q} \end{pmatrix} = 0 \tag{3.33}$$

Two parallel lines

Consider two parallel lines given by $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $\mathbf{a}_2 + \mu\mathbf{b}$.

The normal vector $\mathbf{n}$ to the plane is given by $\mathbf{n} \coloneqq (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}$. From eq. (3.31),

$$(\mathbf{r} - \mathbf{a}_1) \cdot ((\mathbf{a}_2 - \mathbf{a}_1)) \times \mathbf{b}) = 0 \implies \det\begin{pmatrix} \mathbf{r} - \mathbf{a}_1 & \mathbf{a}_2 - \mathbf{a}_1 & \mathbf{q} \end{pmatrix} = 0 \tag{3.34}$$

Normal form

Consider a plane having a normal vector $\mathbf{n} \propto \langle l, m, n \rangle$ and being at a distance of d from the origin. A point $\mathbf{r} \coloneqq \langle x, y, z \rangle$ on the plane is given by,

$$\mathbf{r} \cdot \mathbf{n} = d \tag{3.35}$$

$$\implies lx + my + nz = d \tag{3.36}$$

Intercept form

A plane in three dimensional space having intercepts of a, b and c on the x, y and z axes respectively is given by,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{3.37}$$

Distances from a plane

- The distance of a point $\mathbf{a} \coloneqq \langle x_1, y_1, z_1 \rangle$ from a plane $\Pi \, : \, \mathbf{r} \cdot \mathbf{n} = d$ is given by,

  $$\operatorname{dist}(\mathbf{a}, \Pi) = \left| \frac{\mathbf{a} \cdot \mathbf{n} - d}{\|\mathbf{n}\|} \right| \tag{3.38}$$

  If $n \coloneqq \langle a, b, c \rangle$, then by eq. (3.32) and eq. (3.38),

  $$\operatorname{dist}(\mathbf{a}, \Pi) = \left| \frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}} \right| \tag{3.39}$$

- The distance between the parallel planes $\Pi_1 \, : \, \mathbf{r} \cdot \mathbf{n} = d_1$ and $\Pi_1 \, : \, \mathbf{r} \cdot \mathbf{n} = d_2$ is given by,

  $$\operatorname{dist}(\Pi_1, \Pi_2) = \frac{d_1 \sim d_2}{\|\mathbf{n}\|} \tag{3.40}$$

Family of planes

The family of planes passing through the line of intersection of two planes Π_1 and Π_2 is given by,

$$\Pi_1 + \lambda\Pi_2 = 0, \; \lambda \in \mathbb{R} \tag{3.41}$$

The bisector plane between two intersecting planes given by $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by,

$$\left| \frac{\mathbf{r} \cdot \mathbf{n}_1 - d_1}{\|n\|_1} \right| = \left| \frac{\mathbf{r} \cdot \mathbf{n}_2 - d_2}{\|\mathbf{n}_2\|} \right| \tag{3.42}$$

Angles made by a plane

- The angle θ between the planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is given by,

$$\theta = \arccos\left(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2\right) = \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_2\|\|\mathbf{n}_2\|}\right) \tag{3.43}$$

- The angle θ between the plane given by $\mathbf{r} \cdot \mathbf{n} = d$ and the line given by $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ is given by,

$$\theta = \arcsin\left(\hat{\mathbf{n}} \cdot \hat{\mathbf{b}}\right) = \arcsin\left(\frac{\mathbf{n} \cdot \mathbf{b}}{\|\mathbf{n}\|\|\mathbf{b}\|}\right) \tag{3.44}$$

Chapter 4

Vector Calculus

4.1 Vector Fields

4.2 Nabla Operator

The operator $\boldsymbol{\nabla}$ is pronounced as "nabla" (or del).

In 2D, it is defined as,

$$\boldsymbol{\nabla} \coloneqq \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \tag{4.1}$$

In 3D, it is defined as,

$$\boldsymbol{\nabla} \coloneqq \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \tag{4.2}$$

In a space with n dimensions, it is defined as,

$$\boldsymbol{\nabla} \coloneqq \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\rangle \tag{4.3}$$

4.3 Gradient

Consider $f(\boldsymbol{x})$ to be a scalar multivariable function.

In 2D, f will be denoted as $f(x, y)$.

In 3D, f will be denoted as $f(x, y, z)$.

In a space with n dimensions, f will be denoted as $f(x_1, x_2, \dots, x_n)$.

$$\boldsymbol{\nabla} f \coloneqq \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \tag{4.4}$$

4.4 Divergence

The divergence of a vector field $\mathbf{F}$ is defined as,

$$\begin{aligned}\nabla \cdot \mathbf{F} &\coloneqq \frac{\partial(\mathbf{F}\cdot\hat{\boldsymbol{e}}_1)}{\partial x_1} + \frac{\partial(\mathbf{F}\cdot\hat{\boldsymbol{e}}_2)}{\partial x_2} + \cdots + \frac{\partial(\mathbf{F}\cdot\hat{\boldsymbol{e}}_n)}{\partial x_n} \\ &= \sum_{i=1}^{n} \frac{\partial(\mathbf{F}\cdot\hat{\boldsymbol{e}}_i)}{\partial x_i}\end{aligned} \tag{4.5}$$

In 2D,

$$\nabla \cdot \mathbf{F} = \frac{\partial(\mathbf{F}\cdot\hat{\mathbf{i}})}{\partial x} + \frac{\partial(\mathbf{F}\cdot\hat{\mathbf{j}})}{\partial y} \tag{4.6}$$

In 3D,

$$\nabla \cdot \mathbf{F} = \frac{\partial(\mathbf{F}\cdot\hat{\mathbf{i}})}{\partial x} + \frac{\partial(\mathbf{F}\cdot\hat{\mathbf{j}})}{\partial y} + \frac{\partial(\mathbf{F}\cdot\hat{\mathbf{k}})}{\partial z} \tag{4.7}$$

Positive divergence acts as a *source* while negative divergence acts as a *sink*.

4.5 Curl

In three dimensions, $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

The curl of a 3D vector field $\mathbf{F}$ is defined as,

$$\begin{aligned}\nabla \times \mathbf{F} &\coloneqq \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{F}\cdot\hat{\mathbf{i}} & \mathbf{F}\cdot\hat{\mathbf{j}} & \mathbf{F}\cdot\hat{\mathbf{k}} \end{vmatrix} \\ &= \left\langle \frac{\partial(\mathbf{F}\cdot\mathbf{k})}{\partial y} - \frac{\partial(\mathbf{F}\cdot\mathbf{j})}{\partial z}, \frac{\partial(\mathbf{F}\cdot\mathbf{i})}{\partial z} - \frac{\partial(\mathbf{F}\cdot\mathbf{k})}{\partial x}, \frac{\partial(\mathbf{F}\cdot\mathbf{j})}{\partial x} - \frac{\partial(\mathbf{F}\cdot\mathbf{i})}{\partial y} \right\rangle\end{aligned} \tag{4.8}$$

4.6 Laplacian

Consider $f(\mathbf{x})$ to be a scalar multivariable function.

In 2D, the Laplacian of f is defined as,

$$\nabla^2 f \coloneqq \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \tag{4.9}$$

In 3D, the Laplacian of f is defined as,

$$\nabla^2 f \coloneqq \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4.10}$$

In a space with n dimensions, the Laplacian of f is defined as,

$$\begin{aligned}\nabla^2 f &\coloneqq \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \\ &= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}\end{aligned} \tag{4.11}$$

4.7 Directional Derivative

The derivative of a scalar multivariable function $\mathrm{f}(\boldsymbol{x})$ in the direction of a vector $\boldsymbol{v}$ is given by,

$$\begin{aligned}\nabla_{\boldsymbol{v}} \mathrm{f}(\boldsymbol{x}) &\coloneqq \lim_{h \to 0} \frac{\mathrm{f}(\boldsymbol{x} + h\hat{\boldsymbol{v}}) - \mathrm{f}(\boldsymbol{x})}{h\|\boldsymbol{v}\|} \\ &= \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \boldsymbol{\nabla} \mathrm{f}(\boldsymbol{x}) = \hat{\boldsymbol{v}} \cdot \boldsymbol{\nabla} \mathrm{f}(\boldsymbol{x}) \end{aligned} \tag{4.12}$$

Chapter 5

Differential Geometry

5.1 Defining Quantities

5.1.1 Vectors

Position

Position is denoted as $\mathbf{r}(t)$.

In 2D, $\mathbf{r}(t) \coloneqq \langle x(t), y(t) \rangle$.

In 3D, $\mathbf{r}(t) \coloneqq \langle x(t), y(t), z(t) \rangle$.

In an n-dimensional space,

$$\mathbf{r}(t) \coloneqq \sum_{i=0}^{n} r_i(t) \hat{\mathbf{e}}_i \tag{5.1}$$

Taylor-Series expansion

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(0) + \frac{d\mathbf{r}}{dt}(0) \cdot t + \frac{d^2\mathbf{r}}{dt^2}(0) \cdot \frac{t^2}{2!} + \frac{d^3\mathbf{r}}{dt^3}(0) \cdot \frac{t^3}{3!} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{d^n\mathbf{r}}{dt^n}(0) \cdot \frac{t^n}{n!} \end{aligned} \tag{5.2}$$

Displacement

Displacement is denoted as $\mathbf{d}(\mathbf{t_1}, \mathbf{t_2})$.

$$\mathbf{d}(t_1, t_2) \coloneqq \mathbf{r}(t_2) - \mathbf{r}(t_1) \tag{5.3}$$

Velocity

Velocity is denoted as $\boldsymbol{v}(t)$.

$$\boldsymbol{v}(t) \coloneqq \frac{d\mathbf{r}}{dt}(t) \tag{5.4}$$

Acceleration

Acceleration is denoted as $\mathbf{a}(t)$.

$$\mathbf{a}(t) \coloneqq \frac{d\boldsymbol{v}}{dt}(t) = \frac{d^2\mathbf{r}}{dt^2}(t) \tag{5.5}$$

Jerk

Jerk is denoted as $\mathbf{j}(t)$.

$$\mathbf{j}(t) \coloneqq \frac{d\mathbf{a}}{dt}(t) = \frac{d^2\boldsymbol{v}}{dt^2}(t) = \frac{d^3\mathbf{r}}{dt^3}(t) \tag{5.6}$$

Tangent

The unit tangent vector is denoted by $\hat{\mathbf{T}}(t)$.

$$\hat{\mathbf{T}}(t) \coloneqq \hat{\boldsymbol{v}}(t) = \frac{\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|} \tag{5.7}$$

Normal

The unit normal vector is denoted by $\hat{\mathbf{N}}(t)$.

$$\hat{\mathbf{N}}(t) \coloneqq \frac{d\hat{\mathbf{T}}(t) \big/ dt}{\left\| d\hat{\mathbf{T}}(t) \big/ dt \right\|} \tag{5.8}$$

Binormal

The unit binormal vector is denoted by $\hat{\mathbf{B}}(t)$.

$$\hat{\mathbf{B}}(t) \coloneqq \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t) \tag{5.9}$$

5.1.2 Scalars

Speed

Speed is denoted as $v(t)$.

$$v(t) \coloneqq \|\boldsymbol{v}(t)\| \tag{5.10}$$

Arc Length

Arc length is denoted as $s(t)$.

$$s(t) \coloneqq \int_0^t v(\tau) d\tau \tag{5.11}$$

Curvature

Curvature is denoted as $\kappa(t)$.

$$\kappa(t) \coloneqq \frac{\|\boldsymbol{v}(t) \times \mathbf{a}(t)\|}{\|\boldsymbol{v}(t)\|^3} \tag{5.12}$$

Radius of Curvature

Radius of curvature is denoted as $R(t)$.

$$R(t) \coloneqq \frac{1}{\kappa(t)} \tag{5.13}$$

Tangential Angle

Tangential angle is denoted as $\varphi(t)$.

$$\varphi(t) \coloneqq \int_0^t \kappa(\tau) \cdot \nu(\tau) d\tau \tag{5.14}$$

Torsion

Torsion is denoted as $\tau(t)$.

$$\tau(t) \coloneqq \frac{\det \begin{pmatrix} \boldsymbol{v}(t) & \boldsymbol{a}(t) & \mathbf{j}(t) \end{pmatrix}}{\|\boldsymbol{v}(t) \times \boldsymbol{a}(t)\|^2} \tag{5.15}$$

If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\tau(t) \coloneqq \frac{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2} \tag{5.16}$$

5.2 Theorems

$$\boxed{\begin{bmatrix} \mathbb{R} & \mathfrak{R} \\ g & \infty \end{bmatrix}}$$

The End.